CURVATURE INVARIANTS, DIFFERENTIAL OPERATORS AND LOCAL HOMOGENEITY

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ABSTRACT. We first prove that a Riemannian manifold (M, g) with globally constant additive Weyl invariants is locally homogeneous. Then we use this result to show that a manifold (M, g) whose Laplacian commutes with all invariant differential operators is a locally homogeneous space.

1. INTRODUCTION

A Riemannian manifold (M, g) is said to be locally homogeneous if its pseudogroup of local isometries acts transitively on (M, g). Of course, every homogeneous space is locally homogeneous, but the converse does not hold. Moreover, there exist examples of locally homogeneous spaces which are not locally isometric to a globally homogeneous space (see [10], [18], [21] for explicit examples).

There are several special classes of Riemannian manifolds where only locally homogeneous examples are known. For example, this happens for the classes of harmonic spaces, spaces with volume-preserving geodesic symmetries, and ball-homogeneous spaces. It remains an interesting open problem if there exist examples in these classes which are not locally homogeneous. (See [3], [11], [12], [22] for a survey and further references about these spaces.)

In order to attack this problem, it is important to have at our disposal a series of necessary and sufficient conditions for local homogeneity. The main purpose of this paper is to provide such a criterion. It stems from the fact that in the study of the special Riemannian manifolds cited above, and in many other places, the scalar curvature invariants play a fundamental role. These invariants are by definition polynomials in the components of the Riemannian curvature tensor and its covariant derivatives which do not depend on the choice of orthonormal basis of the tangent space at each point. It is clear that such invariants are global constants on a locally homogeneous space. We will prove that the converse holds. In fact, we will show more, and prove that the converse also holds when we restrict to the special set formed by the additive Weyl invariants. (See below for the definition.) Therefore, we study the Weyl invariants in more detail and prove that each of them can be represented as a product of additive ones.

The second part of the paper contains an application and concerns differential operators on a Riemannian manifold which are invariant under isometries. It originates from the fact that an (M, g) has volume-preserving geodesic symmetries if its Laplacian Δ commutes with a whole class of special invariant differential operators, namely the class of Euclidean Laplacians. In this paper we will prove that (M, g) is locally homogeneous if Δ commutes with a special set of invariant second order differential operators constructed by means of additive Weyl invariants. As a consequence of this result, it turns out that a Riemannian manifold whose Laplacian commutes with all invariant differential operators is a locally homogeneous space with volume-preserving geodesic symmetries.

2. Curvature invariants and local homogeneity

Let (M, g) be an n-dimensional, smooth, connected Riemannian manifold and let $OM = (OM, \pi, M, O(n))$ be the bundle of orthonormal frames over (M, g). An element $u = (q; u_1, ..., u_n)$ of OM induces an isometry, also denoted by u, of $V = \mathbb{R}^n$ on $T_q M$ defined by

$$u(\xi) = u(\xi^1, ..., \xi^n) = \sum_{i=1}^{n} \xi^i u_i.$$

Let ∇ denote the Levi-Civita connection and R the corresponding Riemannian curvature tensor of (M, g). For each $s \geq 0$, the tensor field $\nabla^s R$ defines a map K^s from OM to $\otimes^{s+4} V^*$ given by

$$K^s(u)(\xi_1, ..., \xi_{s+4}) = \nabla^s R_{|\pi(u)}(u\xi_1, ..., u\xi_{s+4}),$$

where $\xi_1, ..., \xi_{s+4} \in V$.

The orthogonal group $O(n)$ acts on the left on the tensor product $\otimes^{\ell} V^*$ by the rule

$$(aT)(\xi_1, ..., \xi_{\ell}) = T(a^{-1}\xi_1, ..., a^{-1}\xi_{\ell})$$

and it is easy to verify that K^s is an equivariant map with respect to the actions of $O(n)$ on OM and on $\otimes^{s+4} V^*$, that is,

$$K^s(ua) = a^{-1} K^s(u)$$

for all $u \in OM$ and $a \in O(n)$.

Following [17] (see also [13]) we consider now the map $\phi_m : OM \rightarrow W_m = \bigoplus_{s=0}^{m}(\otimes^{s+4} V^*)$ defined by

$$\phi_m(u) = \sum_{s=0}^{m} K^s(u).$$

Then ϕ_m is also equivariant, that is,

$$\phi_m(ua) = a^{-1}\phi_m(u)$$

for all $u \in OM, a \in O(n)$.

Next, let $\mathbb{R}[W]^{O(n)}$ denote the $\mathbb{R}$-algebra of $O(n)$-invariant polynomial functions on a representation space W. If $p(w)$ is such a polynomial for $W = W_m$, then $p \circ \phi_m$ is constant along the fibers of OM :

$$(p \circ \phi_m)(ua) = p(a^{-1}\phi_m(u)) = p(\phi_m(u)) = (p \circ \phi_m)(u).$$

Therefore, there exists a unique C^∞ map $f : M \to \mathbb{R}$ such that

(2.1) $$f \circ \pi = p \circ \phi_m,$$

that is, the following diagram commutes:

$$
\begin{array}{ccc}
OM & \xrightarrow{\phi_m} & W_m \\
\pi\downarrow & & \downarrow p \\
M = OM/O(n) & \xrightarrow{f} & \mathbb{R}.
\end{array}
$$

We call f a *scalar curvature invariant of order m* of (M, g). Note that f is an $O(n)$-invariant polynomial in the components of $R, \nabla R, ..., \nabla^m R$ with respect to some fixed orthonormal basis of each tangent space $T_q M, q \in M$.

Since $O(n)$ is compact, we have the following result of Hilbert [14], [15], [24] :

Proposition 2.1. *The algebra $\mathbb{R}[W]^{O(n)}$ is finitely generated. If $p_1, ..., p_t$ are generators, then the polynomial map $p : W \to \mathbb{R}^t, w \mapsto p(w) = (p_1(w), ..., p_t(w))$ separates the orbits of $O(n)$ in W.*

This means that w and w' are in the same orbit if and only if $p(w) = p(w')$.
As an immediate consequence we have:

Corollary 2.2. *Let (M, g) be a Riemannian manifold such that all the scalar curvature invariants of order m are global constants. Then $\phi_m(OM)$ is contained in a single $O(n)$-orbit in W_m.*

Proof. $p \circ \phi_m$ is constant on the fibers in OM, and (2.1) shows that it does not depend on the points q of M, either.

Note that for $m = 0$ we obtain that (M, g) is *curvature homogeneous.*
Now, we prove

Theorem 2.3. *Let (M, g) be a Riemannian manifold of dimension n such that all scalar curvature invariants of order $m+1$ with $m+1 \le \frac{n(n-1)}{2}$ are constant. Then (M, g) is locally homogeneous. Moreover, (M, g) is uniquely determined by these curvature invariants up to a local isometry.*

Proof. ϕ_{m+1} is contained in a single orbit for all $m + 1 \le \frac{n(n-1)}{2}$. This implies that (M, g) is infinitesimally homogeneous, since the Singer integer $k(p)$ at a point $p \in M$ is not greater than $\frac{1}{2}n(n-1) - 1$ [17]. As a consequence, (M, g) is locally homogeneous (see [17] for the global homogeneous case and [13] for a local version). The last statement follows from Theorem 2.5 in [13].

Note that the converse is trivially true. Hence we obtain a criterion for local homogeneity. Note also that another estimate for the Singer integer is given in [8, p. 165] without proof, namely $k(p) + 1 < \frac{3}{2}n$.

3. Weyl and additive Weyl invariants

Now we focus our attention on special scalar curvature invariants. It follows from Weyl's theory of invariants [24] (see also [1, p. 76]) that the invariant polynomials are contractions in the components of the curvature tensor and its covariant

derivatives, and hence, linear combinations of Weyl invariants. Here a (non-trivial) *Weyl invariant W* (i.e., a product of traces [1]) is given by

$$(3.1) \qquad\qquad W = \text{tr}(\nabla^{\ell_1} R \otimes \cdots \otimes \nabla^{\ell_\nu} R)$$

$(\ell_j \in \mathbb{N}_0, j = 1, ..., \nu; \nu \in \mathbb{N}), \nabla^0 R = R$ and tr means the complete trace with respect to some permutation of the indices.

Obviously, any Weyl invariant W given by (3.1) is defined for every Riemannian manifold (M, g). For two Weyl invariants W_1 and W_2 we say $W_1 = W_2$ if and only if $W_1(q) = W_2(q)$ for each $q \in M$ and every (M, g). Now, let $\lambda \in \mathbb{R}_+$. Then we have $W(\lambda g) = \lambda^{-\mu} W(g)$, and we call $\mu \in \mathbb{N}$ the *degree* of W. So, for W in (3.1) we get $\deg W = \frac{1}{2}(\ell_1 + \cdots + \ell_\nu) + \nu$. Then we have for $W = W_1 W_2$ that $\deg W = \deg W_1 + \deg W_2$.

Clearly, we have

Corollary 3.1. *Let (M, g) be a Riemannian manifold with globally constant Weyl invariants. Then (M, g) is locally homogeneous.*

Now, as usual, we write $T_{i_1 \dots i_k} = T(e_{i_1}, ..., e_{i_k})$ for a tensor of type $(0, k)$ with respect to an orthonormal basis $(e_1, ..., e_n)$ of $T_q M$ at $q \in M$. Then a Weyl invariant W at q may be written, with respect to this frame, in the following form:

$$(3.2) \qquad W = \sum_{(a_1)_1, ..., (a_\nu)_{\ell_\nu + 4}} \sigma(\delta^{A_1 \cdots A_\nu})(\nabla^{\ell_1} R)_{A_1} \cdots (\nabla^{\ell_\nu} R)_{A_\nu},$$

where $A_\kappa = ((a_\kappa)_1, ..., (a_\kappa)_{\ell_\kappa + 4}), (\nabla^{\ell_\kappa} R)_{A_\kappa} = \nabla^{\ell_\kappa}_{(a_\kappa)_1 \cdots (a_\kappa)_{\ell_\kappa}} R_{(a_\kappa)_{\ell_\kappa + 1} \cdots (a_\kappa)_{\ell_\kappa + 4}}, \kappa = 1, ..., \nu$, and $\sigma(\delta^{A_1 \cdots A_\nu}) = \delta^{\sigma(a_1)_1 \sigma(a_1)_2} \dots \delta^{\sigma(a_\nu)_{\ell_\nu + 3} \sigma(a_\nu)_{\ell_\nu + 4}}$, δ^{ij} being the Kronecker symbol and σ a permutation of the indices $(a_1)_1, ..., (a_\nu)_{\ell_\nu + 4}$.

Then we have

Lemma 3.2. *Let W be a Weyl invariant of degree μ. Then $W = W_1 W_2$, where $W_i, i = 1, 2$, are Weyl invariants of degree $\mu_i < \mu$, if and only if there exists a permutation σ in the expression (3.2) of W which restricts to a permutation of a proper subset $\{A_{\alpha_1}, ..., A_{\alpha_r}\}$ of $\{A_1, ..., A_\nu\}$.*

Proof. Let $W = W_1 W_2$ with degree $W_i < \mu$. Then the result is obvious.

Conversely, suppose we can restrict σ in such a way. Put $W_1 = \text{tr} \nabla^{\ell_{\alpha_1}} R \otimes \cdots \otimes \nabla^{\ell_{\alpha_r}} R$ with the permutation σ_1 in (3.2) given by the restriction of σ to $\{A_{\alpha_1}, ..., A_{\alpha_\nu}\}$. Then the restriction of σ to $\{A_{\beta_1}, ..., A_{\beta_s}\} = \{A_1, ..., A_\nu\} \setminus \{A_{\alpha_1}, ..., A_{\alpha_r}\}$ is also a permutation. Using σ_2 in the expression (3.2), we obtain a second Weyl invariant $W_2 = \text{tr} \nabla^{\ell_{\beta_1}} R \otimes \cdots \otimes \nabla^{\ell_{\beta_s}} R$, and obviously $W = W_1 W_2$. Since $\mu = \mu_1 + \mu_2$ we have $\mu_i < \mu$, $i = 1, 2$, and the result follows.

Next, a Weyl invariant W is said to be *additive* if for all $M = M_1 \times M_2$ we have

$$W_M(q_1, q_2) = W_{M_1}(q_1) + W_{M_2}(q_2),$$

where $(q_1, q_2) \in M_1 \times M_2$ and where W_M, W_{M_1} and W_{M_2} are the corresponding Weyl invariants of W for M, M_1 and M_2, respectively. For example, it is well-known that the scalar curvature τ is an additive Weyl invariant, but τ^2 is not additive.

Now we prove

Theorem 3.3. *Every Weyl invariant is a product of additive Weyl invariants.*

Proof. Let $(M = M_1 \times M_2, g)$ be the Riemannian product of (M_1, g_1) and (M_2, g_2) and denote by $\nabla, \overset{(1)}{\nabla}, \overset{(2)}{\nabla}$ their respective Levi-Civita connections and by $R, \overset{(1)}{R}, \overset{(2)}{R}$ the corresponding curvature tensors. Let dim $M_1 = n_1$ and dim $M_2 = n_2$, and put $I_1 = \{1, ..., n_1\}, I_2 = \{n_1 + 1, ..., n_1 + n_2\}$. With a similar notation as for (3.2) we have

$$(3.3) \qquad (\nabla^\ell R)_{A_\ell}(q) = \begin{cases} (\overset{(1)}{\nabla}{}^\ell \overset{(1)}{R})_{A_\ell}(q_1) & \text{for } A_\ell \subseteq I_1 \times \cdots \times I_1, \\ (\overset{(2)}{\nabla}{}^\ell \overset{(2)}{R})_{A_\ell}(q_2) & \text{for } A_\ell \subseteq I_2 \times \cdots \times I_2, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $q_1 \in M_1, q_2 \in M_2$ and $\{a_1, ..., a_{n_1}\}$, respectively $\{a_{n_1+1}, ..., a_{n_1+n_2}\}$, is an orthonormal basis of $T_{q_1}M$, respectively $T_{q_2}M_2$.

For a Weyl invariant W we have the following two possibilities:

(i) $W = V_1 V_2$ where $\mu_i = \text{degree } V_i < \mu = \text{degree } W, i = 1, 2$;
(ii) W cannot be decomposed as such a product.

In case (ii), Lemma 3.2 yields that the possible permutation σ in the representation (3.2) of W cannot restrict to a proper subset of $(A_1, ..., A_\nu)$. But then it follows at once from (3.2) and (3.3) that in the corresponding sum we have only terms with indices belonging either to I_1 or to I_2, that is, $W = W_1 + W_2$ and W is additive. In case (i) we may make a similar consideration for V_1 and V_2 as for W. After a finite number of steps we get $W = \bar{V}_1 \cdots \bar{V}_r$, $\mu = \mu_1 + \cdots + \mu_r, 1 \leq \mu_i < \mu$, where each $\bar{V}_\alpha, \alpha = 1, ..., r$, cannot be decomposed further. Then, as in case (ii), it follows that each $\bar{V}_\alpha$ is additive, and so the proof is complete.

From Corollary 3.1 and Theorem 3.3 we get at once

Corollary 3.4. *Let (M, g) be a Riemannian manifold such that each additive Weyl invariant is globally constant. Then (M, g) is locally homogeneous.*

Finally, we give a criterion for the additivity of a Weyl invariant by means of the irreducibility of the polynomials $p \in \mathbb{R}[W]^{O(n)}$. We start with

Lemma 3.5. *Let f be a scalar curvature invariant. Then $f = f_1 f_2$ with scalar curvature invariants $f_i, i = 1, 2$, if and only if there exists a reducible polynomial for f in the representation (2.1).*

Proof. First, let $f \circ \pi = p \circ \phi_m$ with $p = p_1 p_2$. This gives

$$f \circ \pi = (p_1 p_2) \circ \phi_m = (p_1 \circ \phi_m)(p_2 \circ \phi_m) = (f_1 \circ \pi)(f_2 \circ \pi) = (f_1 f_2) \circ \pi$$

and hence $f = f_1 f_2$.

Conversely, let $f = f_1 f_2$. Put $f_1 \circ \pi = p_1 \circ \phi_{m_1}, f_2 \circ \pi = p_2 \circ \phi_{m_2}$. Then we can find suitable $\tilde{p}_1, \tilde{p}_2, \tilde{m}$ such that $f_1 \circ \pi = \tilde{p}_1 \circ \phi_{\tilde{m}}, f_2 \circ \pi = \tilde{p}_2 \circ \phi_{\tilde{m}}$; and we get

$$f \circ \pi = (f_1 f_2) \circ \pi = (f_1 \circ \pi)(f_2 \circ \pi) = (\tilde{p}_1 \circ \phi_{\tilde{m}})(\tilde{p}_2 \circ \phi_{\tilde{m}}) = (\tilde{p}_1 \tilde{p}_2) \circ \phi_{\tilde{m}}.$$

Therefore we can choose $p = \tilde{p}_1 \tilde{p}_2$ for f in the representation (2.1).

From this we get

Corollary 3.6. *Let W be a Weyl invariant. If all possible polynomials p in $W \circ \pi = p \circ \phi_m$ are irreducible, then W is additive.*

Proof. The hypothesis implies that W cannot be equal to $W_1 W_2$ where W_1 and W_2 are Weyl invariants. Then the result follows from Theorem 3.3.

To prove the converse of this corollary, and hence to obtain the required criterion, we shall use three lemmas which we consider first.

Lemma 3.7. *Let $W = \sum_{i=1}^{\ell} a_i W_i$ be an additive Weyl invariant, where the a_i are all non-zero and the W_i are linearly independent Weyl invariants. Then each W_i is additive.*

Proof. Following Theorem 3.3 we put $W_i = \prod_{j=1}^{\kappa_i} U_{ij}, i = 1, ..., \ell$, where each U_{ij} is an additive Weyl invariant. Hence, we have

$$W = \sum_{i=1}^{\ell} a_i \prod_{j=1}^{\kappa_i} U_{ij}.$$

Now, let $(q, q) \in M \times M$ where M is an arbitrary Riemannian manifold. Then we have

$$W_{M \times M}(q, q) = 2 W_M(q) = \sum_{i=1}^{\ell} a_i \prod_{j=1}^{\kappa_i} U_{ij M \times M}(q, q) = \sum_{i=1}^{\ell} a_i 2^{\kappa_i} \prod_{j=1}^{\kappa_i} U_{ij M}(q),$$

and thus

$$W = \sum_{i=1}^{\ell} a_i 2^{\kappa_i - 1} \prod_{j=1}^{\kappa_i} U_{ij}$$

or

$$\sum_{i=1}^{\ell} a_i (2^{\kappa_i - 1} - 1) W_i = 0.$$

The hypotheses now imply that $\kappa_i = 1$ for $i = 1, ..., \ell$, and hence $W_i = U_{i1}$, where U_{i1} is additive.

Lemma 3.8. *Let W be a non-vanishing additive Weyl invariant such that $W = \prod_{j=1}^{\kappa} U_j$ where each U_j is a Weyl invariant. Then $\kappa = 1$.*

Proof. It follows from Theorem 3.3 that $U_j = \prod_{i=1}^{\kappa_j} U_{ji}$ where each U_{ji} is an additive Weyl invariant. Thus, we have

$$W = \prod_{j=1}^{\kappa} U_{j1} \cdots U_{j\kappa_j}.$$

Next, using the same procedure as in the proof of Lemma 3.7, we now get

$$2W = 2^{\sum_{j=1}^{\kappa} \kappa_j} W$$

and so, $\sum_{j=1}^{\kappa} \kappa_j = 1$. The result now follows at once since $\kappa_j \geq 1, j = 1, ..., \kappa$.

Lemma 3.9. *Let $W = \prod_{i=1}^{k} f_i$ be an additive Weyl invariant where each $f_i, i = 1, \dots, k$, is a scalar curvature invariant. Then $\kappa = 1$.*

Proof. Since every scalar curvature invariant is a sum of Weyl invariants, we may put

$$f_i = \sum_{\ell_i=1}^{\mu_i} a_{i\ell_i} U_{i\ell_i}$$

where the $U_{i\ell_i}$ are Weyl invariants. This yields

(3.4) $$W = \sum_{\ell_1=1}^{\mu_1} \cdots \sum_{\ell_k=1}^{\mu_k} a_{1\ell_1} \cdots a_{k\ell_k} U_{1\ell_1} \cdots U_{k\ell_k}.$$

Now, if some of the products $U_{1\ell_1}, ..., U_{k\ell_k}$ are linearly dependent, then we may always write

$$W = \sum \cdots \sum \tilde{a}_{1\ell_1} \cdots \tilde{a}_{k\ell_k} U_{1\ell_1} \cdots U_{k\ell_k}$$

where we sum here only over the linearly independent products. Then Lemma 3.7 implies that each of these products is an additive Weyl invariant, and hence all the products in (3.4) are also additive. Then Lemma 3.8 implies that each of these products can only contain one factor, and so $\mu_1 = \mu_2 = \cdots = \mu_k = k = 1$.

From these lemmas we obtain

Corollary 3.10. *Let W be an additive Weyl invariant. Then all the possible polynomials p in $W \circ \pi = p \circ \phi_m$ are irreducible.*

Proof. Put $p = p_1 \cdots p_k$, where each $p_i, i = 1, \ldots, k$, is irreducible, and $p \circ \phi_m = (p_1 \circ \phi_m) \cdots (p_k \circ \phi_m) = (f_1 \circ \pi) \cdots (f_k \circ \pi)$. Then $W = f_1 \cdots f_k$, and Lemma 3.9 yields $k = 1$, which proves the required result.

4. COMMUTATIVITY OF INVARIANT DIFFERENTIAL OPERATORS

Before giving an application of these results we start with some motivating considerations. Let $q \in (M, g)$ and let ξ be a unit vector of $T_q M$. $\gamma : t \mapsto \exp_m(t\xi)$ denotes the geodesic through q with tangent vector ξ at q. We always suppose $t < i(q)$, where $i(q)$ denotes the injectivity radius at q. Now, let $\{e_1, ..., e_n\}$ be an orthonormal basis of $T_q M$ and $(x_1, ..., x_n)$ the corresponding system of normal coordinates centered at q and with $\frac{\partial}{\partial x_i}(q) = e_i, i = 1, ..., n$. Further, let θ_q be the volume density function of $\exp_q$ given by

$$\theta_q = (\det(g_{ij}))^{1/2},$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Then the local geodesic symmetry $s_q : \exp_q(t\xi) \mapsto \exp_q(-t\xi)$ is a volume-preserving (up to sign) local diffeomorphism if and only if

$$\theta_q(\exp_q(t\xi)) = \theta_q(\exp_q(-t\xi))$$

for all unit $\xi \in T_q M$ and all $q \in M$. The study of these spaces has been initiated in [6], and such spaces have been called *D'Atri spaces* [23]. We refer to [11] for an extensive survey. They may also be characterized by using a commutativity relation for some special differential operators. More precisely, define a local differential operator $\bar{\Delta}_q$ by

$$\bar{\Delta}_q f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

and put

$$\tilde{\Delta}^{(k)} f(q) = (\bar{\Delta}_q)^k f(q).$$

Then $\tilde{\Delta}^{(k)}$ is a global differential operator on M, called the *Euclidean Laplacian of order* $2k$ [7]. They can be expressed as a function of Δ^k, the curvature tensor and its covariant derivatives (see [7] for more details). We have [20]: a) (M, g) is a D'Atri space if the Laplacian Δ commutes with all $\tilde{\Delta}^{(k)}, k \in \mathbb{N}$, and b) (M, g) is a D'Atri space if and only if Δ commutes with all differential operators $f \mapsto \tilde{\Delta}^{(k)}(\theta_q f)$. (See also [16].)

Now we prove

Theorem 4.1. *Let* (M, g) *be a Riemannian manifold such that the Laplacian* Δ *commutes with all second order differential operators* $L = \eta^{ij} \nabla^2_{ij}$, *where* η *is a tensor field such that* η^{ij} *are polynomials in the components of* R *and its covariant derivatives and such that* $\alpha = \eta^i_i$ *is an additive Weyl invariant. Then* (M, g) *is locally homogeneous.*

Proof. Following [19], $L\Delta = \Delta L$ implies $\nabla^2_{ij} \alpha = 0$ and moreover, if M is locally irreducible, then α is a global constant. Hence the result follows from Corollary 3.4. Next, let M be locally reducible and put $M = M_1 \times \cdots \times M_r$. Then $\nabla^2_{AB} \alpha = \nabla^2_{AB}(\alpha_1 + \cdots + \alpha_r)$ since $\alpha = \alpha_1 + \cdots + \alpha_r$, where $\alpha_i, i = 1, ..., r$, are the corresponding Weyl invariants on M_i. This implies $\nabla^2_{ab} \alpha_i = 0$, where a, b are indices on M_i. Since M_i is irreducible, we get that α_i is constant on $M_i, i = 1, ..., r$, and so α is a global constant. Hence, the result follows again from Corollary 3.4.

We note that, given an additive Weyl invariant α, then $L = \frac{1}{n} \alpha g^{ij} \nabla^2_{ij} = \frac{\alpha}{n} \Delta$ is a second order differential operator as in Theorem 4.1, although there are several other possibilities as one can see immediately from (3.2). One just deletes one of the δ in the summation to produce other examples. By using a similar proof one then obtains

Corollary 4.2. *Let* (M, g) *be a Riemannian manifold such that the Laplacian* Δ *commutes with all differential operators* $\alpha \Delta$, *where* α *is an additive Weyl invariant. Then* (M, g) *is locally homogeneous.*

Further, we have

Corollary 4.3. *Let* (M, g) *be a Riemannian manifold and denote by* $\text{Isom}_{\text{loc}}(M)$ *the pseudogroup of local isometries of* M. *Further, let* $\mathcal{D}_{\text{loc}}(M)$ *denote the algebra of all* $\text{Isom}_{\text{loc}}(M, g)$*-invariant differential operators. Let* $\mathcal{Z}$ *be its center. Then if* $\Delta \in \mathcal{Z}$ *we have*

  (i) (M, g) *is locally homogeneous;*
 (ii) (M, g) *is a D'Atri space;*
(iii) (M, g) *is a* $\mathcal{C}$*-space.*

Here a $\mathcal{C}$-*space* is an (M, g) such that the eigenvalues of the Jacobi operators $R_{\gamma'.} \gamma'$ are constant along the unit speed geodesic γ for each γ [4].

Proof. (i) follows from Theorem 4.1. (ii) follows from [16], [20] because the Euclidean Laplacians $\tilde{\Delta}^{(k)}$ (and also the operators $\tilde{\Delta}^{(k)}(\theta_q \cdot)$) belong to $\mathcal{D}_{loc}$. Finally, (iii) follows from a result of Sumitomo as in [4].

Corollary 4.4. *Let (M, g) be a four-dimensional Einstein space with the property that Δ commutes with all second order operators as in Theorem 4.1 (or Corollary 4.2), where α is an additive Weyl invariant of order $m + 1 \leq 6$. Then (M, g) is locally symmetric.*

Proof. It follows from Theorem 4.1 (or Corollary 4.2) and Theorem 2.3 that (M, g) is locally homogeneous. Then a well-known result of Jensen implies that (M, g) is locally symmetric.

A *commutative* (in the broader sense) Riemannian manifold is a homogeneous (M, g) such that $\mathcal{D}$ is a commutative algebra (see [3], [11] for more details). Hence we have

Corollary 4.5. *A complete, simply connected (M, g) such that $\mathcal{D}$ is a commutative algebra is a homogeneous space.*

Moreover, we have

Corollary 4.6. *Let (M, g) be a complete, simply connected Riemannian manifold of dimension 2 or 3 or an Einstein space when $\dim M = 4$. Then (M, g) is a commutative space if and only if Δ belongs to the center of $\mathcal{D}$.*

Proof. First we note that two- or three-dimensional, complete, simply connected spaces are D'Atri spaces if and only if they are commutative [6], [9]. Then, the required result follows from this and Corollary 4.3. For four-dimensional Einstein spaces the result follows at once from Corollary 4.4 since symmetric spaces are commutative spaces.

Note that (M, g) is locally isometric to a commutative space in Corollary 4.6 when we delete the words "complete" and "simply connected" in the hypothesis.
We finish with

Corollary 4.7. *There exist D'Atri spaces such that $\Delta \notin \mathcal{Z}$.*

Proof. The non-symmetric harmonic spaces discovered by Damek and Ricci [5] are D'Atri spaces but are not $\mathcal{C}$-spaces [2], [3]. Hence, Corollary 4.3 implies that Δ does not belong to $\mathcal{Z}$.

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